February 3, 1887.

Professor STOKES, D.C.L., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "On the Waves produced by a Single Impulse in Water of any Depth, or in a Dispersive Medium." By Sir W. THOMSON, Knt., LL.D., F.R.S. Received January 26, 1887.

For brevity and simplicity consider only the case of *two-dimensional motion.*

All that it is necessary to know of the medium is the relation between the wave-velocity and the wave-length of an endless procession of periodic waves. The result of our work will show us that the velocity of progress of a zero, or maximum, or minimum, in any part of a varying group of waves, is equal to the velocity of progress of periodic waves of wave-length equal to a certain length, which may be defined as the wave-length in the neighbourhood of the particular point looked to in the group (a length which will generally be intermediate between the distances from the point considered to its next-neighbour corresponding points on its two sides).

Let $f(m)$ denote the velocity of propagation corresponding to wave-length $2\pi/\lambda$. The Fourier-Cauchy-Poisson synthesis gives

$$u=\int_0^\infty dm\cos m[x-tf(m)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the effect at place and time (x, t) of an infinitely intense disturbance at place and time $(0, 0)$. The principle of interference as set forth by Prof. Stokes and Lord Rayleigh in their theory of group-velocity and wave-velocity suggests the following treatment for this integral:—

When $x-tf(m)$ is very large, the parts of the integral (1) which lie on the two sides of a small range, $\mu-\alpha$ to $\mu+\alpha$, vanish by annulling interference; μ being a value, or the value, of m, which makes

$$\frac{d}{dm}\{m[x-tf(m)]\}=0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that we have $$x = t\{f(\mu)+\mu f'(\mu)\} = yt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{3}$$

where $$y = f(\mu)+\mu f'(\mu)\,;^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{4}$$

and we have by Taylor's theorem for $m-\mu$ very small:

$$m[x-tf(m)] = \mu[x-tf(\mu)]-\tfrac{1}{2}t[\mu f''(\mu)+2f'(\mu)](m-\mu)^2, \quad . \tag{5}$$

or, modifying by (3)

$$m[x-tf(m)] = t\{\mu^2 f'(\mu)+\tfrac{1}{2}[-\mu f''(\mu)-2f'(\mu)](m-\mu)^2\}. \quad . \tag{6}$$

Put now $$m-\mu = \frac{\sigma\sqrt{2}}{t^{\frac{1}{2}}[-\mu f''(\mu)-2f'(\mu)]^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \tag{7}$$

and using the result in (1), we find

$$u = \frac{\sqrt{2}\int_{-\infty}^{\infty} d\sigma \cos[t\mu^2 f'(\mu)+\sigma^2]}{t^{\frac{1}{2}}[-\mu f''(\mu)-2f'(\mu)]^{\frac{1}{2}}}\,; \quad . \quad . \quad . \quad . \quad . \quad . \tag{8}$$

the limits of the integral being here $-\infty$ to ∞, because the denominator of (7) is so infinitely great that, though $\pm\alpha$, the arbitrary limits of $m-\mu$, are infinitely small, α multiplied by it is infinitely great.

Now we have $$\int_{-\infty}^{\infty} d\sigma \cos\sigma^2 = \int_{-\infty}^{\infty} d\sigma \sin\sigma^2 = \sqrt{(\tfrac{1}{2}\pi)} \quad . \quad . \quad . \quad . \quad . \tag{9}$$

Hence (8) becomes

$$u = \frac{\cos[t\mu^2 f'(\mu)]-\sin[t\mu^2 f'(\mu)]}{t^{\frac{1}{2}}[-\mu f''(\mu)-2f'(\mu)]^{\frac{1}{2}}} = \frac{\sqrt{2}\cos[t\mu^2 f'(\mu)+\frac{1}{4}\pi]}{t^{\frac{1}{2}}[-\mu f''(\mu)-2f'(\mu)]^{\frac{1}{2}}}. \tag{10}$$

To prove the law of wave-length and wave-velocity for any point of the group, remark that, by (3)

$$t\mu^2 f'(\mu) = \mu[x-tf(\mu)],$$

and therefore the numerator of (10) is equal to $\sqrt{2}\cos\theta$, where

$$\theta = \mu[x-tf(\mu)]+\tfrac{1}{4}\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{10'}$$

and by (2) and (3) $$\frac{d}{d\mu}\{\mu[x-tf(\mu)]\} = 0;$$

by which we see that

$$d\theta/dx = \mu, \quad \text{and} \quad d\theta/dt = -\mu f(\mu), \quad . \quad . \quad . \quad . \quad . \tag{10'}$$

which proves the proposition.

* This is the group-velocity according to Lord Rayleigh's generalisation of Prof. Stokes's original result.

Example (1).—As a first example take deep-sea waves; we have

$$f(m) = \sqrt{\frac{g}{m}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which reduces (4), (3), and (10) to

$$y = \tfrac{1}{2}\sqrt{\frac{g}{\mu}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$x = \tfrac{1}{2}\sqrt{\frac{g}{\mu}} \cdot t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$u = \frac{1}{g\sqrt{2}} \frac{t}{x^{\frac{3}{2}}} \left(\cos \frac{gt^2}{4x} + \sin \frac{gt^2}{4x}\right) = \frac{t}{gx} \cos \left(\frac{gt^2}{4x} - \frac{\pi}{4}\right), \quad . \quad (14)$$

which is Cauchy and Poisson's result for places where x is very great in comparison with the wave-length $2\pi/\mu$, that is to say, for place and time such that $gt^2/4x$ is very large.

Example (2).—Waves in water of depth D:

$$f(m) = \sqrt{\left\{\frac{g}{m} \frac{1-\epsilon^{-2m\text{D}}}{1+\epsilon^{-2m\text{D}}}\right\}}. \quad . \quad . \quad . \quad . \quad (15)$$

Example (3).—Light in a dispersive medium.

Example (4).—Capillary gravitational waves:

$$f(m) = \sqrt{\left(\frac{g}{m} + \text{T}m\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Example (5).—Capillary waves:

$$f(m) = \sqrt{(\text{T}m)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Example (6).—Waves of flexure running along a uniform elastic rod:

$$f(m) = m\sqrt{\frac{\text{B}}{w}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where B denotes the flexural rigidity, and w the mass per unit of length.

These last three examples have been taken by Lord Rayleigh as applications of his generalisation of the theory of group-velocity; and he has pointed out in his "Standing Waves in Running Water" (London Mathematical Society, December 13, 1883) the important peculiarity of Example (4) in respect to the critical wave-length which gives minimum wave-velocity, and therefore group-velocity equal to wave-velocity. The working out of our present problem for this case, or any case in which there are either minimums or maximums, or both maximums and minimums, of wave-velocity, is particularly interest-

ing, but time does not permit its being included in the present communication.

For Examples (5) and (6) the denominator of (10) is imaginary; and the proper modification, from (7) forwards, gives for these and such cases, instead of (14), the following :—

$$u = \frac{\cos\left[t\mu^2 f'(\mu)\right] + \sin\left[t\mu^2 f'(\mu)\right]}{t^{\frac{1}{2}}\left[\mu f''(\mu) + 2f'(\mu)\right]^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

The result is easily written down for each of the two last cases [Examples (5) and (6)].

II. "On the Formation of Coreless Vortices by the Motion of a Solid through an inviscid incompressible Fluid." By Sir W. THOMSON, Knt., LL.D., F.R.S. Received February 1, 1887.

Take the simplest case: let the moving solid be a globe, and let the fluid be of infinite extent in all directions. Let its pressure be of any given value, P, at infinite distances from the globe, and let the globe be kept moving with a given constant velocity, V.

If the fluid keeps everywhere in contact with the globe, its velocity relatively to the globe at the equator (which is the place of greatest relative velocity) is $\frac{3}{2}$V. Hence, unless $P > \frac{5}{8}V^2$,* the fluid will not remain in contact with the globe.

Suppose, in the first place, P to have been $> \frac{5}{8}V^2$, and to be suddenly reduced to some constant value $< \frac{5}{8}V^2$. The fluid will be thrown off the globe at a belt of a certain breadth, and a violently disturbed motion will ensue. To describe it, it will be convenient to speak of velocities and motions *relative to the globe.* The fluid must, as indicated by the arrow-heads in fig. 1, flow partly backwards and partly forwards, at the place, I, where it impinges on the globe, after having shot off at a tangent at A. The back-flow along the belt that had been bared must bring to E some fluid; and the free surface of this fluid must collide with the surface of the fluid leaving the globe at A. It might be supposed that the result of this collision would be a "vortex sheet," which in virtue of its instability, would get drawn out and mixed up indefinitely, and be carried away by the fluid farther and farther from the globe. A definite amount of kinetic energy would be *practically annulled* in a manner which I hope to explain in an early communication to the Royal Society of Edinburgh.

But it is impossible, either in our ideal inviscid incompressible

* The density of the fluid is taken as unity.